A coupled variational principle for 2D interactions between water-waves and a rigid-body containing fluid

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New variational principles are given for the two-dimensional interactions between gravity driven water-waves and a rotating and translating rectangular vessel dynamically coupled to its interior potential flow with uniform vorticity. The complete set of equations of motion for the exterior water waves, and the exact nonlinear hydrodynamic equations of motion for the vessel in the roll/pitch, sway/surge, and heave directions, and also the full set of equations of motion for the vessel's interior fluid, relative to the body coordinate system attached to the rotating-translating vessel, are derived from two Lagrangian functionals.

1. Introduction

Luke (1967) presented a variational principle for the classical water-wave problem described by equations

$$\begin{aligned}
&\Delta\Phi := \Phi_{XX} + \Phi_{YY} = 0 \quad \text{for} \quad -H(X) < Y < \Gamma(X,t)\,,\\
&\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY = 0 \quad \text{on} \quad Y = \Gamma(X,t)\,,\\
&\Phi_Y = \Gamma_t + \Phi_X\Gamma_X \quad \text{on} \quad Y = \Gamma(X,t)\,,\\
&\Phi_Y + \Phi_X H_X = 0 \quad \text{on} \quad Y = -H(X)\,,
\end{aligned} \tag{1.1}$$

where (X,Y) is the spatial coordinate system, $\Phi(X,Y,t)$ is the velocity potential of an irrotational fluid lying between $Y=-H(X)$ and $Y=\Gamma(X,t)$ with the gravity acceleration g acting in the negative Y direction. In the horizontal direction X, the fluid domain is cut off by a vertical surface Σ which extends from the bottom to the free surface. Then Luke's variational principle reads

$$\delta\mathcal{L}(\Phi,\Gamma) \;=\; \delta\int_{t_1}^{t_2}\int_{X_1}^{X_2}\int_{-H(X)}^{\Gamma(X,t)} -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY\right)dY\,dX\,dt = 0\,, \tag{1.2}$$

with variations in $\Phi(X,Y,t)$ and $\Gamma(X,t)$ subject to the restrictions $\delta\Phi=0$ at the end points of the time interval, t_1 and t_2. In (1.2) the gradient vector field is denoted by ∇, and ρ is the water density.

Miloh (1984) presented an extension of Luke's variational principle for water waves interacting with several bodies on or below a free surface which oscillate at a common frequency. van Daalen (1993) and van Daalen and van Groesen and Zandbergen (1993), hereafter DGZ, extended the Hamiltonian formulation of surface waves due to

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Zakharov (1968), Broer (1974), and Miles (1977) to water waves in hydrodynamic interaction with freely floating bodies starting from a variational principle of the form

$$\delta\mathcal{L}=\delta\int_{t_1}^{t_2}\int_{\Omega(t)}-\rho\left(\Phi_t+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi+gY\right)d\Omega dt+\delta\int_{t_1}^{t_2}\left(\mathsf{KE}^{vessel}-\mathsf{PE}^{vessel}\right)dt=0\,, \tag{1.3}$$

where KE^{vessel} is the kinetic energy of the vessel and PE^{vessel} is the potential energy of the vessel. In this action integral the system under consideration, $\Omega(t)$, consists of a fluid, bounded by the impermeable bottom $Y=-H(X)$, the free surface $Y=\Gamma(X,t)$, and the wetted surface S of a rigid-body. In the horizontal direction X, the fluid domain is cut off by a vertical surface Σ at $X=X_1,X_2$ which extends from the bottom to the free surface. DGZ used this Lagrangian action to derive the complete set of equations of motion- i.e. equations (1.1) and the hydrodynamic equations of motion for the rigid-body. But they did not present the exact nonlinear equations for the rigid-body motion, due to the approximation in their definition for the body angular velocity in KE^{vessel}. Compare the second term in equation (3) of DGZ with the third term in the last line of equation (1.4). Also the second term in the last line of equation (1.4) is absent in equation (3) of DGZ. van Groesen and Andonowati (2017) presented a Boussinesq-type Hamiltonian formulation for wave-ship interactions.

The variational principle presented by DGZ was for an *empty* rigid-body in hydrodynamic interaction with exterior water-waves. But in the present article, in order to take into account the coupled dynamics between fluid sloshing in a vessel while in an ambient wave field, with coupling to the vessel motion, the second part of the variational principle (1.3) should be modified to include the kinetic and potential energies of the interior fluid. To do this, we first present the general form of a three-dimensional Lagrangian action for a rigid-body with interior fluid motion, and then in §3 we show that how a reduced two-dimensional version of this functional can be derived for the purposes of this paper. Alemi Ardakani (2010) derived the exact form of a three-dimensional Lagrangian action for a rigid-body containing fluid which undergoes 3D rotational and translational motions. The action integral takes the form

$$\begin{aligned}\mathcal{L}(\omega,\mathbf{q})&=\int_{t_1}^{t_2}\int_{\Omega'}\left(\mathsf{KE}^{fluid}-\mathsf{PE}^{fluid}\right)d\Omega' dt+\int_{t_1}^{t_2}\left(\mathsf{KE}^{vessel}-\mathsf{PE}^{vessel}\right)dt\\&=\int_{t_1}^{t_2}\int_{\Omega'}\Big(\tfrac{1}{2}\|\dot{\mathbf{x}}\|^2+\dot{\mathbf{x}}\cdot\left(\omega\times(\mathbf{x}+\mathbf{d})+\mathbf{Q}^T\dot{\mathbf{q}}\right)+\mathbf{Q}^T\dot{\mathbf{q}}\cdot\left(\omega\times(\mathbf{x}+\mathbf{d})\right)+\tfrac{1}{2}\|\dot{\mathbf{q}}\|^2\\&\quad+\tfrac{1}{2}\omega\cdot\Big(\|\mathbf{x}+\mathbf{d}\|^2\mathbf{I}-(\mathbf{x}+\mathbf{d})\otimes(\mathbf{x}+\mathbf{d})\Big)\omega-g\left(\mathbf{Q}(\mathbf{x}+\mathbf{d})+\mathbf{q}\right)\cdot\mathbf{E}_2\Big)\rho d\Omega' dt\\&\quad+\int_{t_1}^{t_2}\Big(\tfrac{1}{2}m_v\|\dot{\mathbf{q}}\|^2+(\omega\times m_v\overline{\mathbf{x}}_v)\cdot\mathbf{Q}^T\dot{\mathbf{q}}+\tfrac{1}{2}\omega\cdot\mathbf{I}_v\omega-m_v g\left(\mathbf{Q}\overline{\mathbf{x}}_v+\mathbf{q}\right)\cdot\mathbf{E}_2\Big)dt\,,\end{aligned} \tag{1.4}$$

where the body frame which is attached to the moving rigid-body has coordinates $\mathbf{x}=(x,y,z)$, the distance between the centre of rotation and the origin of the body frame is $\mathbf{d}=(d_1,d_2,d_3)$, the fluid-tank system has a uniform translation $\mathbf{q}(t)=(q_1,q_2,q_3)$ relative to the spatial frame $\mathbf{X}=(X,Y,Z)$, integral is over the volume Ω' of the interior fluid, $\otimes$ denotes the tensor product, $\mathbf{I}$ is the 3×3 identity matrix, $\mathbf{I}_v$ is the dry vessel mass moment of inertia relative to the point of rotation, m_v is the mass of the dry vessel, and $\overline{\mathbf{x}}_v=(\overline{x}_v,\overline{y}_v,\overline{z}_v)$ is the centre of mass of the dry vessel relative to the body frame, $\mathbf{E}_2$ is the unit vector in the Y direction, and $\omega(t)=(\omega_1(t),\omega_2(t),\omega_3(t))$ is the *body* *angular velocity* vector with entries determined from the rotation tensor $\mathbf{Q}(t)$ by

$$\mathbf{Q}^T\dot{\mathbf{Q}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} := \widehat{\omega}\,.$$

The convention for the entries of the skew-symmetric matrix $\widehat{\omega}$ is such that $\widehat{\omega}\mathbf{r} = \omega\times\mathbf{r}$ for any $\mathbf{r}\in\mathbb{R}^3$. The relation between the spatial displacement and the body displacement, and the relation between the body velocity and the space velocity are, respectively

$$\mathbf{X} = \mathbf{Q}\left(\mathbf{x}+\mathbf{d}\right)+\mathbf{q} \quad \text{and} \quad \dot{\mathbf{X}} = \mathbf{Q}\left(\dot{\mathbf{x}}+\omega\times\left(\mathbf{x}+\mathbf{d}\right)+\mathbf{Q}^T\dot{\mathbf{q}}\right)\,. \tag{1.5}$$

This formulation is consistent with the theory of rigid-body motion, where an arbitrary motion can be described by the pair $(\mathbf{Q}(t), \mathbf{q}(t))$. The exact equations of motion for the rigid-body can be derived from the Lagrangian action (1.4).

The interest in this paper is to derive a coupled variational principle for two-dimensional interactions between water-waves and a floating rectangular vessel with interior fluid motion which gives the exact nonlinear Euler-Lagrange equations for the coupled dynamics. The vessel is free to undergo roll/pitch motion (θ), sway/surge motion (q_1), and heave motion (q_2) which are rotation about the centre of rotation in the $Z-$direction relative to rest keel, horizontal displacement along the $X-$axis, and vertical displacement along the $Y-$axis, respectively. It is shown in §3 that adding Luke's Lagrangian action (1.2) to a two-dimensional variant of the Lagrangian action (1.4) gives

$$\begin{aligned}
&\delta\mathcal{L}\left(\Phi,\Gamma,\theta,q_1,q_2\right) = \delta\int_{t_1}^{t_2}\int_{\Omega(t)} -\rho\left(\Phi_t+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi+gY\right)d\Omega dt \\
&+\delta\int_{t_1}^{t_2}\Bigg(\int_0^L\int_0^{\eta(x,t)}\left[\tfrac{1}{2}\left(\phi_x^2+\phi_y^2\right)+\phi_x\left(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta\right)+\tfrac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)\right. \\
&\qquad \left.+\phi_y\left(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta\right)-g\left(\sin\theta\left(x+d_1\right)+\cos\theta\left(y+d_2\right)+q_2\right)\right]\rho dydx \\
&\qquad +\tfrac{1}{2}m_v\left(\dot{q}_1^2+\dot{q}_2^2\right)-m_v\overline{y}_v\dot{\theta}\left(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta\right)+m_v\overline{x}_v\dot{\theta}\left(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta\right) \\
&\qquad +\tfrac{1}{2}m_v\left(\overline{x}_v^2+\overline{y}_v^2\right)\dot{\theta}^2-m_vg\left(\overline{x}_v\sin\theta+\overline{y}_v\cos\theta+q_2\right)\Bigg)dt = 0\,,
\end{aligned} \tag{1.6}$$

where $\Omega(t)$ is defined as in (1.3), L is the length of the vessel, $\eta(x,t)$ is the fluid height relative to the body coordinate system (x,y) which is attached to the moving vessel, $(\overline{x}_v, \overline{y}_v)$ is the centre of mass of the dry vessel relative to the body frame, m_v is the mass of the dry vessel, (d_1, d_2) is the distance between the centre of rotation and the origin of the body frame, and $\phi(x,y,t)$ is the velocity potential for the interior fluid motion, yet to be determined. See Figure 1 for a sketch of the coordinate systems. The second part of (1.6) is kinetic and potential energy of all domain, for the fluid in the vessel in moving coordinates such that the extra fictitious forces emerge. The Lagrangian action (1.6) can be used for derivation of the set of equations of motion for the classical water-wave problem (1.1) and also the hydrodynamic equations of motion for the rigid-body in the roll/pitch, sway/surge, and heave directions.

The second variational principle is a variant of Luke's variational principle (1.2) for the vessel's interior rotating-translating fluid motion. It is shown in §2 that the complete set of equations for the interior fluid motion relative to the body coordinate system can

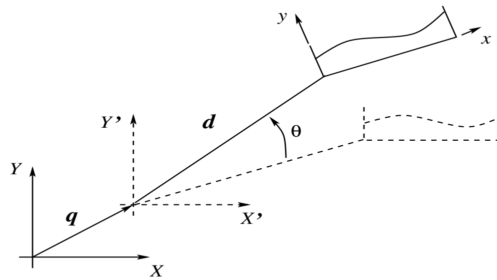

FIGURE 1. Diagram showing coordinate systems for rotating-translating vessel. The coordinate system (X', Y') is the translation by $\mathbf{q} = (q_1, q_2)$ of the fixed coordinate system (X, Y). The distance from the centre of rotation to the origin of the body coordinate system (x, y) is $\mathbf{d} = (d_1, d_2)$.

be obtained from the Lagrangian action

$$\begin{aligned}\delta\mathcal{L}(\phi,\eta) \;=\; \delta\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} & \Big[-\Big(\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \dot\theta\,(y+d_2)\,\phi_x - \dot\theta\,(x+d_1)\,\phi_y\Big) \\ & +\tfrac{1}{2}\,(x+d_1)\,(-g\sin\theta - \ddot q_1\cos\theta - \ddot q_2\sin\theta) \\ & +\tfrac{1}{2}\,(y+d_2)\,(-g\cos\theta + \ddot q_1\sin\theta - \ddot q_2\cos\theta)\Big]\,\rho\,dy\,dx\,dt = 0\,. \end{aligned} \tag{1.7}$$

The paper starts with derivation of a variational principle for the fluid motion in a rotating-translating rectangular vessel in §2. In §3 a variational principle is presented for interactions between the exterior water-waves and the rigid-body containing fluid. The exact nonlinear hydrodynamic equations for the rigid-body motion are derived. The paper ends with concluding remarks in §4.

2. A variational principle for the interior fluid motion

The configuration of the fluid in a rotating-translating vessel is shown in Figure 1. The fluid occupies the region $0 \leqslant y \leqslant \eta(x,t)$ with $0 \leqslant x \leqslant L$. The two-dimensional Euler equations relative to a rotating-translating coordinate system (x, y) given by ALEMI ARDAKANI AND BRIDGES (2012) are

$$\begin{aligned}\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= -g\sin\theta + 2\dot\theta v + \ddot\theta\,(y+d_2) + \dot\theta^2\,(x+d_1) - \ddot q_1\cos\theta - \ddot q_2\sin\theta\,,\\ \frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= -g\cos\theta - 2\dot\theta u - \ddot\theta\,(x+d_1) + \dot\theta^2\,(y+d_2) + \ddot q_1\sin\theta - \ddot q_2\cos\theta\,,\\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0\,,\end{aligned} \tag{2.1}$$

where $Du/Dt = u_t + uu_x + vu_y$. The velocity field (u, v) is relative to the body frame and p is the pressure field. Relative to the body frame, the boundary conditions are

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L\,, \quad v = 0 \quad \text{at} \quad y = 0\,, \tag{2.2}$$

and

$$p = 0 \quad \text{and} \quad \eta_t + u\eta_x = v\,, \quad \text{at} \quad y = \eta(x,t)\,, \tag{2.3}$$

where the surface tension is neglected in the boundary condition for the pressure. The vorticity $\mathcal{V} = v_x - u_y$ satisfies the equation $D\mathcal{V}/Dt = -2\ddot\theta$.

Now we introduce a velocity potential $\phi(x,y,t)$ such that

$$u(x,y,t) = \phi_x + \dot{\theta}(y+d_2) \quad \text{and} \quad v(x,y,t) = \phi_y - \dot{\theta}(x+d_1)\,. \tag{2.4}$$

The velocity field in (2.4) satisfies the vorticity equation. The vorticity is constant in space and satisfies $\mathcal{V} = -2\dot{\theta}$. Substitution of (2.4) into the continuity equation in (2.1) leads to Laplace's equation for $\phi(x,y,t)$

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad 0 \leqslant y \leqslant \eta(x,t)\,, \quad 0 \leqslant x \leqslant L\,, \tag{2.5}$$

and substitution of (2.4) into the momentum equations in (2.1) leads to Bernoulli's equation for the pressure field

$$\begin{aligned}&\frac{p}{\rho} + \phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_y^2\right) - \tfrac{1}{2}(x+d_1)(-g\sin\theta - \ddot{q}_1\cos\theta - \ddot{q}_2\sin\theta)\\ &-\tfrac{1}{2}(y+d_2)(-g\cos\theta + \ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta) + \dot{\theta}(y+d_2)\phi_x - \dot{\theta}(x+d_1)\phi_y = Be(t)\,,\end{aligned} \tag{2.6}$$

where $Be(t)$ is the Bernoulli function which can be absorbed into $\phi(x,y,t)$. Therefore the dynamic free surface boundary condition in (2.3) at $y = \eta(x,t)$ becomes

$$\begin{aligned}&\phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \dot{\theta}(\eta+d_2)\phi_x - \tfrac{1}{2}(x+d_1)(-g\sin\theta - \ddot{q}_1\cos\theta - \ddot{q}_2\sin\theta)\\ &\qquad -\tfrac{1}{2}(\eta+d_2)(-g\cos\theta + \ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta) - \dot{\theta}(x+d_1)\phi_y = 0\,.\end{aligned} \tag{2.7}$$

In terms of the velocity potential $\phi(x,y,t)$, the kinematic free surface boundary condition in (2.3) becomes

$$\eta_t + \left(\phi_x + \dot{\theta}(\eta+d_2)\right)\eta_x = \phi_y - \dot{\theta}(x+d_1) \quad \text{at} \quad y = \eta(x,t)\,, \tag{2.8}$$

and the rigid-wall boundary conditions in (2.2) become

$$\phi_x = -\dot{\theta}(y+d_2) \quad \text{at} \quad x = 0, L\,, \quad \phi_y = \dot{\theta}(x+d_1) \quad \text{at} \quad y = 0\,. \tag{2.9}$$

Following Luke, the variational principle for the interior rotating-translating fluid is

$$\delta\mathcal{L}(\phi,\eta) = \delta\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} p(x,y,t)\,dydxdt = 0\,. \tag{2.10}$$

Now substituting for $p(x,y,t)$ from (2.6) into (2.10) gives (1.7). Taking the variations in (1.7) gives

$$\begin{aligned}\delta\mathcal{L}(\phi,\eta) &= \int_{t_1}^{t_2}\int_0^L \left(\eta_t + \eta_x\phi_x - \phi_y + \eta_x\dot{\theta}(\eta+d_2) + \dot{\theta}(x+d_1)\right)\delta\phi\big|^{y=\eta}\rho dxdt\\ &+ \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)}(\phi_{xx}+\phi_{yy})\,\delta\phi\rho dydxdt - \int_{t_1}^{t_2}\int_0^L\left(\nabla\phi\cdot\mathbf{n} + \dot{\theta}(x+d_1)\right)\delta\phi\big|_{y=0}\rho dxdt\\ &+ \int_{t_1}^{t_2}\int_0^\eta\left(-\nabla\phi\cdot\mathbf{n} + \dot{\theta}(y+d_2)\right)\delta\phi\big|_{x=0}\rho dydt\\ &+ \int_{t_1}^{t_2}\int_0^\eta\left(-\nabla\phi\cdot\mathbf{n} - \dot{\theta}(y+d_2)\right)\delta\phi\big|_{x=L}\rho dydt + \int_{t_1}^{t_2}\int_0^L p(x,\eta,t)\,\delta\eta dxdt = 0\,.\end{aligned} \tag{2.11}$$

A detailed derivation of (2.11) is given in Appendix A. From (2.11) it is obvious that invariance of $\mathcal{L}$ with respect to a variation in the free surface elevation η yields the dynamic free surface boundary condition (2.7). Similarly, the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ yields the field equation (2.5). Also the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ at $y = 0$, $x = 0$, and $x = L$

recovers the rigid wall boundary conditions in (2.9). And the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ at $y = \eta$ recovers the kinematic free surface boundary condition (2.8).

3. A variational principle for the exterior water-waves and the rigid-body motion containing fluid

The complete set of differential equations for the exterior water-waves interacting with the rigid-body in the plane can be obtained from a variant of the Lagrangian action (1.3) which takes the form

$$\begin{aligned}\delta\mathcal{L} &= \delta\int_{t_1}^{t_2}\int_{\Omega(t)} -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY\right)d\Omega dt\\ &+\delta\int_{t_1}^{t_2}\int_{\Omega'}\left(\mathsf{KE}^{fluid} - \mathsf{PE}^{fluid}\right)d\Omega' dt + \delta\int_{t_1}^{t_2}\left(\mathsf{KE}^{vessel} - \mathsf{PE}^{vessel}\right)dt = 0\,,\end{aligned} \tag{3.1}$$

where the second line in (3.1) is a 2D variant of the Lagrangian action (1.4), which can be obtained by substituting $\dot{\mathbf{x}} = (u, v, 0) = \left(\phi_x + \dot\theta\,(y + d_2)\,, \phi_y - \dot\theta\,(x + d_1)\,, 0\right)$, $\mathbf{q} = (q_1, q_2, 0)$, $\omega = \left(0, 0, \dot\theta\right)$, $\mathbf{d} = (d_1, d_2, 0)$, $\overline{\mathbf{x}}_v = (\overline{x}_v, \overline{y}_v, 0)$, and

$$\mathbf{Q} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix},$$

into (1.4). Then the variational principle (3.1) reduces to (1.6).

In order to take the variations in (1.6), variational Reynold's transport theorem should be used, since the domain of integration Ω is time-dependent. See Flanders (1973), Daniliuk (1976), and Gagarina and van der Vegt and Bokhove (2013) for the background mathematics on variational analogue of Reynold's transport theorem. Then according to the usual procedure in the calculus of variations, (1.6) for all variations in the free surface elevation Γ, the velocity potential Φ, the vessel position $\mathbf{q} = (q_1, q_2)$ and the vessel orientation θ becomes

$$\begin{aligned}&\delta\mathcal{L}\left(\Phi, \Gamma, \theta, q_1, q_2\right) = \int_{t_1}^{t_2}\int_{X_1}^{X_2} -\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY\right)\Big|^{Y=\Gamma}\rho\delta\Gamma dX dt\\ &+\int_{t_1}^{t_2}\int_S P\left(X, Y, t\right)\left(\delta\mathbf{X}_s\cdot\mathbf{n}\right)ds dt - \int_{t_1}^{t_2}\int_{\Omega(t)}\left(\delta\Phi_t + \nabla\Phi\cdot\nabla\delta\Phi\right)\rho d\Omega dt\\ &+\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)}\left[\phi_x\left(\delta\dot q_1\cos\theta - \dot q_1\sin\theta\delta\theta + \delta\dot q_2\sin\theta + \dot q_2\cos\theta\delta\theta\right)\right.\\ &\qquad +\phi_y\left(-\delta\dot q_1\sin\theta - \dot q_1\cos\theta\delta\theta + \delta\dot q_2\cos\theta - \dot q_2\sin\theta\delta\theta\right)\\ &\qquad \left.+\left(\dot q_1\delta\dot q_1 + \dot q_2\delta\dot q_2\right) - g\left(\cos\theta\left(x + d_1\right)\delta\theta - \sin\theta\left(y + d_2\right)\delta\theta + \delta q_2\right)\right]\rho dy dx dt\\ &+\int_{t_1}^{t_2}\Big(m_v\left(\dot q_1\delta\dot q_1 + \dot q_2\delta\dot q_2\right) - m_v\overline{y}_v\delta\dot\theta\left(\dot q_1\cos\theta + \dot q_2\sin\theta\right)\\ &\qquad -m_v\overline{y}_v\dot\theta\left(\delta\dot q_1\cos\theta - \dot q_1\sin\theta\delta\theta + \delta\dot q_2\sin\theta + \dot q_2\cos\theta\delta\theta\right)\\ &+m_v\overline{x}_v\delta\dot\theta\left(-\dot q_1\sin\theta + \dot q_2\cos\theta\right) + m_v\overline{x}_v\dot\theta\left(-\delta\dot q_1\sin\theta - \dot q_1\cos\theta\delta\theta + \delta\dot q_2\cos\theta - \dot q_2\sin\theta\delta\theta\right)\\ &\qquad +m_v\left(\overline{x}_v^2 + \overline{y}_v^2\right)\dot\theta\delta\dot\theta - m_v g\left(\overline{x}_v\cos\theta\delta\theta - \overline{y}_v\sin\theta\delta\theta + \delta q_2\right)\Big)dt = 0\,,\end{aligned} \tag{3.2}$$

where

$$P(X,Y,t) = -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY\right) \quad \text{on} \quad S\,, \tag{3.3}$$

and noting that these variations are subject to the restrictions that they vanish at the end points of the time interval and on the vertical boundary at infinity Σ. In (3.2) $\mathbf{X}_s$ denotes the position of a point on the wetted vessel surface S relative to the spatial frame (X,Y), and $\mathbf{n}$ is the unit normal vector along $\partial\Omega \supset S$. The change in $\mathbf{X}_s$ due to variations in $\mathbf{q}$ and θ is given by

$$\delta\mathbf{X}_s = \mathbf{Q}'\mathbf{x}_s\delta\theta + \delta\mathbf{q} \quad \text{with} \quad \mathbf{Q}' = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.4}$$

where $\mathbf{x}_s$ is the position of a point on the wetted vessel surface relative to the body frame (x,y). Taking into account the motion of $\Omega(t)$ we may write

$$\begin{aligned} -\frac{d}{dt}\int_{t_1}^{t_2}\int_{\Omega}\delta\Phi\rho d\Omega dt &= -\int_{t_1}^{t_2}\int_{X_1}^{X_2}\Gamma_t\delta\Phi\Big|^{Y=\Gamma}\rho dX dt - \int_{t_1}^{t_2}\int_S\left(\dot{\mathbf{X}}_s\cdot\mathbf{n}\right)\delta\Phi\rho ds dt \\ &\quad -\int_{t_1}^{t_2}\int_{\Omega}\delta\Phi_t\rho d\Omega dt\,. \end{aligned}$$

This is the same as the variational Reynold's transport theorem but with variational derivatives replaced by time derivatives. Noting that the left-hand side vanishes due to the restriction $\delta\Phi = 0$ at times $t = t_1$ and $t = t_2$ this expression simplifies to

$$-\int_{t_1}^{t_2}\int_{\Omega}\delta\Phi_t\rho d\Omega dt = \int_{t_1}^{t_2}\int_{X_1}^{X_2}\Gamma_t\delta\Phi\Big|^{Y=\Gamma}\rho dX dt + \int_{t_1}^{t_2}\int_S\left(\dot{\mathbf{X}}_s\cdot\mathbf{n}\right)\delta\Phi\rho ds dt\,. \tag{3.5}$$

With Green's first identity we may write

$$\begin{aligned} \int_{t_1}^{t_2}\int_{\Omega}\nabla\Phi\cdot\nabla\delta\Phi\rho d\Omega dt &= -\int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho d\Omega dt + \int_{t_1}^{t_2}\int_{\partial\Omega}\left(\nabla\Phi\cdot\mathbf{n}\right)\delta\Phi\rho ds dt \\ &= -\int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho d\Omega dt + \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(-\Gamma_X\Phi_X + \Phi_Y\right)\delta\Phi\Big|^{Y=\Gamma}\rho dX dt \\ &\quad + \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(\Phi_X H_X + \Phi_Y\right)\delta\Phi\Big|_{Y=-H}\rho dX dt + \int_{t_1}^{t_2}\int_S\frac{\partial\Phi}{\partial n}\delta\Phi\rho ds dt\,. \end{aligned} \tag{3.6}$$

Now using the expressions (3.4), (3.5), and (3.6), and integrating by parts and noting that $\delta\mathbf{q}$ and $\delta\theta$ vanish at the end points of the time interval, the variational principle (3.2) simplifies to

$$\begin{aligned} \delta\mathcal{L}\left(\Phi,\Gamma,\theta,q_1,q_2\right) &= \int_{t_1}^{t_2}\int_{X_1}^{X_2} -\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY\right)\Big|^{Y=\Gamma}\rho\delta\Gamma dX dt \\ &+ \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(\Gamma_t + \Gamma_X\Phi_X - \Phi_Y\right)\delta\Phi\Big|^{Y=\Gamma}\rho dX dt + \int_{t_1}^{t_2}\int_S\left(\dot{\mathbf{X}}_s\cdot\mathbf{n} - \frac{\partial\Phi}{\partial n}\right)\delta\Phi\rho ds dt \\ &- \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(\Phi_X H_X + \Phi_Y\right)\delta\Phi\Big|_{Y=-H}\rho dX dt + \int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho d\Omega dt \\ &+ \int_{t_1}^{t_2}\int_S P(X,Y,t)\left(\mathbf{Q}'\mathbf{x}_s\cdot\mathbf{n}\right)\delta\theta\rho ds dt + \int_{t_1}^{t_2}\int_S P(X,Y,t)\,\mathbf{n}\cdot\delta\mathbf{q}\rho ds dt \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\Bigg(\int_0^L\int_0^\eta\Big(-\phi_{xt}\cos\theta+\phi_x\dot\theta\sin\theta+\phi_{yt}\sin\theta+\phi_y\dot\theta\cos\theta-\ddot q_1\\
&\qquad\qquad+\Big(\phi_x+\dot\theta\,(y+d_2)\Big)\left(\phi_{yx}\sin\theta-\phi_{xx}\cos\theta\right)\\
&\qquad\qquad+\Big(\phi_y-\dot\theta\,(x+d_1)\Big)\left(\phi_{yy}\sin\theta-\phi_{xy}\cos\theta\right)\Big)\rho dydx\\
&\quad-m_v\ddot q_1+m_v\overline{y}_v\left(\ddot\theta\cos\theta-\dot\theta^2\sin\theta\right)+m_v\overline{x}_v\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)\Bigg)\,\delta q_1dt\\
&+\int_{t_1}^{t_2}\Bigg(\int_0^L\int_0^\eta\Big(-\phi_{xt}\sin\theta-\phi_x\dot\theta\cos\theta-\phi_{yt}\cos\theta+\phi_y\dot\theta\sin\theta-\ddot q_2-g\\
&\qquad\qquad-\left(\phi_{xx}\sin\theta+\phi_{yx}\cos\theta\right)\Big(\phi_x+\dot\theta\,(y+d_2)\Big)\\
&\qquad\qquad-\left(\phi_{xy}\sin\theta+\phi_{yy}\cos\theta\right)\Big(\phi_y-\dot\theta\,(x+d_1)\Big)\Big)\rho dydx\\
&-m_v\ddot q_2+m_v\overline{y}_v\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)-m_v\overline{x}_v\left(\ddot\theta\cos\theta-\dot\theta^2\sin\theta\right)-m_vg\Bigg)\,\delta q_2dt\\
&+\int_{t_1}^{t_2}\Bigg(\int_0^L\int_0^\eta\left[\phi_x\left(-\dot q_1\sin\theta+\dot q_2\cos\theta\right)+\phi_y\left(-\dot q_1\cos\theta-\dot q_2\sin\theta\right)\right.\\
&\qquad\left.-g\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)\right]\rho dydx+m_v\overline{y}_v\left(\ddot q_1\cos\theta+\ddot q_2\sin\theta\right)\\
&-m_v\overline{x}_v\left(-\ddot q_1\sin\theta+\ddot q_2\cos\theta\right)-m_v\left(\overline{x}_v^2+\overline{y}_v^2\right)\ddot\theta-m_vg\left(\overline{x}_v\cos\theta-\overline{y}_v\sin\theta\right)\Bigg)\,\delta\theta dt=0\,.
\end{aligned}
$$

From (3.7) we conclude that invariance of $\mathcal{L}$ with respect to a variation in the free surface elevation $\mathit{\Gamma}$ yields the dynamic free surface boundary condition in (1.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential $\mathit{\Phi}$ yields the field equation in $\mathit{\Omega}\,(t)$, invariance of $\mathcal{L}$ with respect to a variation in the velocity potential $\mathit{\Phi}$ at $Y=-H\,(X)$ gives the bottom boundary condition in (1.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential $\mathit{\Phi}$ at $Y=\mathit{\Gamma}\,(X,t)$ gives the kinematic free surface boundary condition in (1.1), and invariance of $\mathcal{L}$ with respect to a variation in the velocity potential $\mathit{\Phi}$ on S gives the contact condition on the vessel wetted surface

$$
\frac{\partial\mathit{\Phi}}{\partial n}=\dot{\mathbf{X}}_s\cdot\mathbf{n}\quad\text{on}\quad S\,. \tag{3.8}
$$

Finally, invariance of $\mathcal{L}$ with respect to variations in q_1, q_2 and θ gives the hydrodynamic equations of motion for the rigid-body in the sway/surge (q_1), heave (q_2), and roll/pitch (θ) directions, respectively, which are

- $$
\begin{aligned}
&\left(m_v+m_f\right)\ddot q_1-\int_0^L\int_0^\eta\Big(-\phi_{xt}\cos\theta+\phi_{yt}\sin\theta+\dot\theta\left(\phi_x\sin\theta+\phi_y\cos\theta\right)\\
&\qquad\qquad+\Big(\phi_x+\dot\theta\,(y+d_2)\Big)\left(\phi_{yx}\sin\theta-\phi_{xx}\cos\theta\right)\\
&\qquad\qquad+\Big(\phi_y-\dot\theta\,(x+d_1)\Big)\left(\phi_{yy}\sin\theta-\phi_{xy}\cos\theta\right)\Big)\rho dydx\\
&-m_v\overline{y}_v\left(\ddot\theta\cos\theta-\dot\theta^2\sin\theta\right)-m_v\overline{x}_v\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)-\int_S P\,(X,Y,t)\,n_1\mathrm{d}s=0\,,
\end{aligned}
$$
- $$
\begin{aligned}
&\left(m_v+m_f\right)\left(g+\ddot q_2\right)+\int_0^L\int_0^\eta\Big(\phi_{xt}\sin\theta+\phi_{yt}\cos\theta+\dot\theta\left(\phi_x\cos\theta-\phi_y\sin\theta\right)\\
&\qquad\qquad+\left(\phi_{xx}\sin\theta+\phi_{yx}\cos\theta\right)\Big(\phi_x+\dot\theta\,(y+d_2)\Big)\\
&\qquad\qquad+\left(\phi_{xy}\sin\theta+\phi_{yy}\cos\theta\right)\Big(\phi_y-\dot\theta\,(x+d_1)\Big)\Big)\rho dydx\\
&-m_v\overline{y}_v\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)+m_v\overline{x}_v\left(\ddot\theta\cos\theta-\dot\theta^2\sin\theta\right)-\int_S P\,(X,Y,t)\,n_2\mathrm{d}s=0\,,
\end{aligned}
$$

- $$\begin{aligned} & m_v\left(\overline{x}_v^2+\overline{y}_v^2\right)\ddot{\theta}-\int_0^L\int_0^\eta\left[\phi_x\left(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta\right)+\phi_y\left(-\dot{q}_1\cos\theta-\dot{q}_2\sin\theta\right)\right.\\ & \left.-g\left(\cos\theta\left(x+d_1\right)-\sin\theta\left(y+d_2\right)\right)\right]\rho dydx-m_v\overline{y}_v\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)\\ & +m_v\overline{x}_v\left(-\ddot{q}_1\sin\theta+\ddot{q}_2\cos\theta\right)+m_vg\left(\overline{x}_v\cos\theta-\overline{y}_v\sin\theta\right)\\ & -\int_S P\left(X,Y,t\right)\left(\mathbf{Q}'\mathbf{x}_s\cdot\mathbf{n}\right)ds=0\,, \end{aligned} \tag{3.9}$$

where $P\left(X,Y,t\right)$ is defined in (3.3), and $m_f=\displaystyle\int_0^L\int_0^\eta\rho dydx$ is independent of time.

In summary, the equations of motion for the exterior water-waves in $\Omega\left(t\right)$ are (1.1) with the contact boundary condition (3.8). The equations of motion for the interior fluid motion are the field equation (2.5) with the boundary conditions (2.7), (2.8), and (2.9) which are dynamically coupled to the hydrodynamic equations of motion for the rigid-body (3.9). The terms including the pressure field $P\left(X,Y,t\right)$ in the q_1,q_2 equations, and also in the θ equation in (3.9) are the forces and moment, respectively, acting on the rigid-body due to the exterior water-waves. Similarly the integral terms including derivatives of $\phi\left(x,y,t\right)$ are the forces and moment acting on the rigid-body due to the interior fluid motion.

4. Concluding remarks

The paper is devoted to the derivation of new variational principles for 2D interactions between water-waves and a rigid-body with interior fluid sloshing. The complete set of equations of motion and boundary conditions for the exterior water-waves, and for the vessel's interior fluid motion relative to the rotating-translating coordinate system attached to the moving vessel, and also the exact nonlinear Euler-Lagrange equations for the rigid-body motion in the sway/surge, heave, and roll/pitch directions, are derived from the presented variational principles. The proposed variational principles are applicable to ocean engineering problems. The exact differential equations can be used for the coupled dynamical analysis of a freely floating ship with water on deck or with interior fluid sloshing in the tanks which interacts with exterior water-waves.

Another interesting application of the presented coupled variational principles is for the dynamical response analysis of floating ocean wave energy converters (WEC) such as the OWEL wave energy converter proposed by Offshore Wave Energy Ltd, a schematic of which can be found on the website `http://www.owel.co.uk/`. OWEL is a floating vessel with variable topography and cross-section, open at one end to capture ocean waves. Once they are trapped, the waves undergo interior fluid sloshing while the vessel is interacting with exterior waves. A rise in the wave height is induced within the duct, mainly due to the internal geometry of the WEC. The wave then creates a seal with the rigid-lid resulting in a moving trapped pocket of air ahead of the wave front which drives the power take off.

The proposed variational principles can be used for mathematical modelling of the pendulum-slosh problem. The rigid-body equation for the pendular motion of a suspended rectangular vessel from a single rigid pivoting rod, partially filled with an inviscid fluid can be derived from a simplified version of the variational principle (1.6). Consider the second part of (1.6) with $q_1=q_2=0$ and take the variation with respect to θ to obtain the rigid-body equation. The complete set of equations of motion for the pendulum's interior fluid can be obtained by setting $q_1=q_2=0$ in (1.7) and taking the variations with respect to ϕ and η.

A direction of great interest is to use variational symplectic methods of Gagarina et al. (2014, 2016), Kalogirou and Bokhove (2016), and Bokhove and Kalogirou (2016) to develop energy preserving numerical solvers for the proposed variational principles (1.6) and (1.7) for interactions between ocean waves and floating structures dynamically coupled to interior fluid sloshing, and also for the variational principle (1.7) for the problem of fluid sloshing in vessels undergoing prescribed rigid-body motion in two-dimensions.

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Appendix A. Proof of the variational principle (2.11)

According to the usual procedure in the calculus of variations, (1.7) becomes

$$\begin{aligned}\delta\mathcal{L}(\phi,\eta) = \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} -\Big(\delta\phi_t + \nabla\phi\cdot\nabla\delta\phi + \dot{\theta}\,(y+d_2)\,\delta\phi_x \\ -\dot{\theta}\,(x+d_1)\,\delta\phi_y\Big)\,\rho dydxdt + \int_{t_1}^{t_2}\int_0^L p\,(x,\eta,t)\,\delta\eta dxdt\,,\end{aligned} \tag{A 1}$$

but

$$\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} -\delta\phi_t\rho dydxdt = \int_{t_1}^{t_2}\int_0^L \eta_t\delta\phi\big|^{y=\eta}\rho dxdt\,,$$

noting that $\delta\phi = 0$ at $t = t_1$ and $t = t_2$. Also

$$\begin{aligned}-\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \dot{\theta}\,(y+d_2)\,\delta\phi_x\rho dydxdt = -\int_{t_1}^{t_2}\int_0^{\eta} \dot{\theta}\,(y+d_2)\,\delta\phi\big|_{x=L}\rho dydt \\ +\int_{t_1}^{t_2}\int_0^{\eta} \dot{\theta}\,(y+d_2)\,\delta\phi\big|_{x=0}\rho dydt + \int_{t_1}^{t_2}\int_0^L \eta_x\dot{\theta}\,(\eta+d_2)\,\delta\phi\big|^{y=\eta}\rho dxdt\,,\end{aligned}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \dot{\theta}\,(x+d_1)\,\delta\phi_y\rho dydxdt = \int_{t_1}^{t_2}\int_0^L \dot{\theta}\,(x+d_1)\,\delta\phi\big|^{y=\eta}\rho dxdt \\ -\int_{t_1}^{t_2}\int_0^L \dot{\theta}\,(x+d_1)\,\delta\phi\big|_{y=0}\rho dxdt\,,\end{aligned}$$

and using Green's first identity

$$\begin{aligned}-\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \nabla\phi\cdot\nabla\delta\phi\rho dydxdt = \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} (\phi_{xx}+\phi_{yy})\,\delta\phi\rho dydxdt \\ -\int_{t_1}^{t_2}\int_0^L (-\eta_x\phi_x+\phi_y)\,\delta\phi\big|^{y=\eta}\rho dxdt - \int_{t_1}^{t_2}\int_0^L \nabla\phi\cdot\mathbf{n}\delta\phi\big|_{y=0}\rho dxdt \\ -\int_{t_1}^{t_2}\int_0^{\eta} \nabla\phi\cdot\mathbf{n}\delta\phi\big|_{x=0}\rho dydt - \int_{t_1}^{t_2}\int_0^{\eta} \nabla\phi\cdot\mathbf{n}\delta\phi\big|_{x=L}\rho dydt\,,\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector along the rigid walls. Hence (A 1) converts to (2.11).

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